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THE DRUCKER STABILITY OF A MATERIAL†

K. I. ROMANOV

Moscow

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An invariant formulation of the criterion for the stability of a material, which enables one to investigate the local stability under arbitrary stress conditions, is obtained on the basis of Drucker's postulate [1]. It is shown that the criterion obtained has an explicit physical meaning at large plastic deformations. A relation is established between the stability parameter and other fundamental parameters of the stressed state. © 2001 Elsevier Science Ltd. All rights reserved.

Drucker's postulate has been used to study the plastic stability of various structural elements [2, 3]. It has been shown [4], for the example of a thin-walled, zero-moment cylindrical shell under combined loading, that it is necessary to distinguish between the conditions of loss of global and local stability, that is, the conditions for the stability of the whole body and of the material respectively. In the case of large plastic deformations and, in particular, in technological processes, it is important to investigate the local stability at the danger points. In this connection, a development of the results which have previously been obtained [4] is given below with the aim of devising a criterion for local stability in the general case of a stressed state.

1. THE STABILITY POSTULATE

According to Drucker's stability postulate, the deformation of a body with time-independent properties under isothermal conditions is stable in the small if the work done in the case of infinitesimal increments in the generalized forces dQ_i for corresponding infinitesimal increments in the generalized displacements dq_i is positive [1, 5], that is

$$\Sigma dQ_i dq_i > 0 \tag{1.1}$$

Drucker's postulate can serve as a basis for developing of a criterion for the stability of any material, which may or may not be in a plastic state. However, the treatment below is restricted to plastic media since, in this case, there are experimental data [6] which confirm the conclusions drawn on the basis of Drucker's postulate.

The above-mentioned postulate enables the stability of the whole body (global stability) to be analysed if, by Q_i in condition (1.1), one understands the external generalized forces. In the case when it is necessary to analyse the local stability of an element of a body, the internal generalized forces have to be used in criterion (1.1).

Note the restricted possibility of using postulate (1.1) when analysing global stability. In particular, it is impossible on the basis of this postulate to predict the moment when loss of shape stability accompanying the compression of thin-walled constructional elements, occurs. For this reason, the use of postulate (1.1) is most justified when investigating the stability of a material in a stressed state of a distinct type when significant geometrical changes occur during plastic changes in shape.

The need to require large deformations is attributable to the fact that, in (1.1), one is dealing with generalized forces, that are different from the stresses themselves which, in this investigation, are Eulerian. The difference is due to the fact that, in addition to the stresses, the dimensions of the infinitesimal element, which has been picked out of the deforming body at a certain instant of the loading, also occur in Q_i.

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2. THE STABILITY CRITERION IN THE PRINCIPAL AXES OF THE STRESSED STATE

At a certain instant during a change in shape, we choose an infinitesimal element from the deforming body. Suppose that 1, 2, 3 are the axes of a Cartesian system of coordinates associated with the element and that $a_i (i = 1, 2, 3)$ are the dimensions of the element along the corresponding axes (Fig. 1).

We introduce the generalized forces acting on the faces of the selected element and the increments of the generalized displacements

$$Q_1 = \sigma_1 a_2 a_3,\ Q_2 = \sigma_2 a_1 a_3,$$
$$Q_3 = \sigma_3 a_1 a_2;\ dq_i = a_i d\varepsilon_i$$

Here, σ_i are the principal Eulerian stresses and $d\varepsilon_i = da_i/a_i$ are quantities which are proportional to the rates of deformation at a given instant of loading.

Hence, in accordance with postulate (1.1), we have

$$(d\sigma_1 + \sigma_1 da_2 / a_2 + \sigma_1 da_3 / a_3)d\varepsilon_1 + (d\sigma_2 + \sigma_2 da_1 / a_1 + \sigma_2 da_3 / a_3)d\varepsilon_2 +$$
$$+(d\sigma_3 + \sigma_3 da_1 / a_1 + \sigma_3 da_2 / a_2)d\varepsilon_3 > 0$$

According to the incompressibility condition

$$\sum da_i / a_i = 0$$

We therefore obtain the stability criterion along the principal axes of the stressed state in the form

$$\sum (d\sigma_i - \sigma_i d\varepsilon_i)d\varepsilon_i > 0 \quad (2.1)$$

Note that this result has been obtained previously in [7]. However, the local stability of a material in the case of different stressed states has not been investigated using this criterion. Also, the relation between the stability parameter of the material, which is introduced below, and the other fundamental parameters of the stressed state has not been previously established.

3. THE STABILITY CRITERION ALONG ARBITRARY CARTESIAN AXES

The subsequent transformation of stability criterion (2.1) is associated with the use of the constitutive equations of a specific continuous medium.

For a basis, we take the Saint-Venant–Levy-Mises constitutive equations [8]

$$d\varepsilon_i = \frac{3\widetilde{d\varepsilon_e}}{2\sigma_e} S_i \quad (3.1)$$

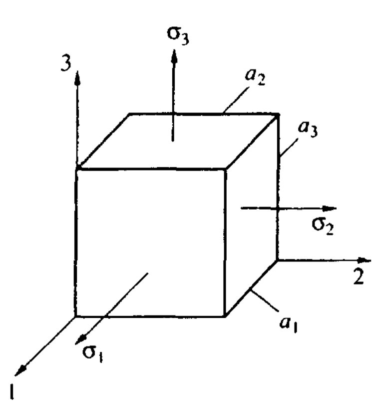

Fig. 1

where $\tilde{d}\varepsilon_e$ and σ_e are equivalent increments in the deformations and the stress, respectively, and S_i are the principal components of the stress deviator.

Taking into account the incompressibility condition of the material $\Sigma d\varepsilon_i = 0$, we transform condition (2.1), using Eqs (3.1), to the form

$$\frac{3\tilde{d}\varepsilon_e}{2\sigma_e}\sum S_i dS_i - \left(\frac{3\tilde{d}\varepsilon_e}{2\sigma_e}\right)^2 \sum S_i^3 - \sigma_0\left(\frac{3\tilde{d}\varepsilon_e}{2\sigma_e}\right)^2 \sum S_i^2 > 0 \tag{3.2}$$

where $\sigma_0 = I_1(\sigma_{ij})/3$ is the mean normal stress and $I_1(\sigma_{ij})$ is the first invariant of the stress tensor.

By definition

$$\sigma_e = [3I_2(S_{ij})]^{1/2} = \left[\frac{3}{2}\sum S_i^2\right]^{1/2}$$

where $I_2(S_{ij})$ is the second invariant of the stress deviator. It is therefore obvious that

$$\sum S_i dS_i = \frac{2}{3}\sigma_e d\sigma_e$$

Moreover, the equality

$$\sum S_i^3 = 3S_1S_2S_3 = 3I_3(S_{ij})$$

follows from the identity $(\Sigma S_j)^3 = 0$.

Stability criterion (3.2) is therefore transformed to the form

$$d\sigma_e\tilde{d}\varepsilon_e - \left(\frac{3\tilde{d}\varepsilon_e}{2\sigma_e}\right)^2 [3I_3(S_{ij}) + 2\sigma_0 I_2(S_{ij})] > 0$$

or

$$\frac{1}{z} - \frac{9}{4[3I_2(S_{ij})]^{3/2}}\left[3I_3(S_{ij}) + \frac{2}{3}I_1(\sigma_{ij})I_2(S_{ij})\right] > 0 \tag{3.3}$$

where

$$\frac{1}{z} = \frac{d\sigma_e}{\sigma_e\tilde{d}\varepsilon_e}$$

and z is a subtangent to the graph of $\sigma_e(\int\tilde{d}\varepsilon_e)$.

At the instant when condition (3.3) is violated, we therefore have the equality $1/z = 1/z_*$ and

$$\frac{1}{z_*} = \frac{\sqrt{3}}{4[I_2(S_{ij})]^{1/2}}\left[3\frac{I_3(S_{ij})}{I_2(S_{ij})} + \frac{2}{3}I_1(\sigma_{ij})\right] \tag{3.4}$$

where z_* is the subtangent to the deformation diagram at the instant of the onset of instability (Fig. 2).

Consequently, the stability criterion of the material for an arbitrary stressed state has the form

$$\Delta = \frac{1}{z} - \frac{1}{z_*} > 0 \tag{3.5}$$

where $1/z_*$ is the stability parameter of the material, which is defined for any stressed state by the invariant representation (3.4).

We will now consider a number of special cases of stressed states in order to illustrate the application of stability criterion (3.5).

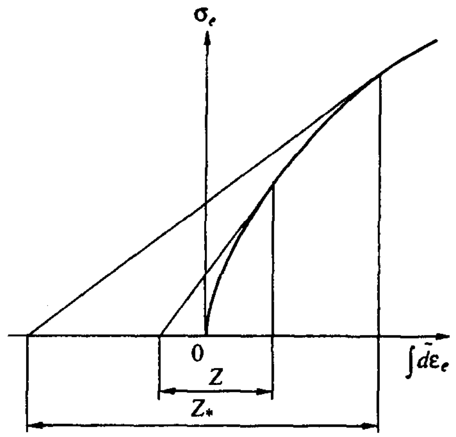

Fig. 2

4. THE SIMPLEST SPECIAL CASES OF A STRESSED STATE

Uniaxial extension. In the case of uniaxial extension, formula (3.4) leads to the equality $1/z_* = 1$, which has been confirmed [5, 8] in experiments on the uniaxial extension of a rod by an axial force up to the formation of a neck.

Uniaxial compression. In the case of uniaxial compression, in accordance with formula (3.4), we have $1/z_* = -1$, and since, under compression, $1/z > 0$, according to criterion (3.5), this means that the process of uniaxial plastic compression is always stable. We have in mind the stability of a material in a stressed state which, in the case under consideration, can be realized within a short sample which is compressed without friction by an axial force. It is well known that, in this case, plastic materials can be deformed without fracture at large deformations.

Simple shear. In the case of simple shear, we have according to formula (3.4), $1/z_* = 0$, that is, $z_* \to \infty$. This, however, does not mean that the material is always stable in the above-mentioned stressed state.

In the case of large plastic deformations, the deformation diagram $\sigma_e - \int \bar{d}\varepsilon_e$ depends to a significant extent [6] on the properties of the material and the type of stressed state. For example, in the case of extension, the graph of the stress against the plastic deformation usually increases up to the fracture of the sample, while, in the case of simple shear, there is usually a maximum in the deformation diagram. In the case of simple shear, the material therefore becomes unstable as $1/z \to 0$ asymptotically since, at this point, $\Delta = 0$.

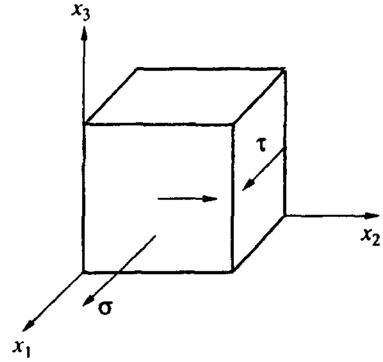

Fig. 3

5. A SIMPLIFIED PLANE STRESSED STATE

In the case of a simplified plane stressed state (Fig. 3) we have, according to formula (3.4)

$$\frac{1}{z_*} = \pm\frac{4 + 15\alpha^2}{4(1 + 3\alpha^2)^{3/2}} \tag{5.1}$$

where $\alpha = \tau/\sigma$ is the loading parameter, the upper sign corresponds to extension with shear $\sigma > 0$ and the lower sign corresponds to compression with shear $\sigma < 0$.

In the special case when $\alpha = 0$, we have $1/z_* = 1$, which corresponds to uniaxial extension and, when $\alpha \to \infty$, we have $1/z_* \to 0$, as in the case of simple shear.

Relation (5.1) has been constructed for the values $\sigma > 0$ in Fig. 4. It has a single extremum at the point (0, 1) and shows that there is a monotone decrease in the parameter $1/z_*$ as the parameter α increases.

We call the graph of $1/z_*$ against the loading parameter the stability diagram of the stressed state of the material. Using this graph, it is possible to interpret stability criterion (3.5) graphically. For example, in the case of a simplified plane stressed state, the values of $1/z$, determined in an actual problem at a certain instant of loading at a given point of a deforming body by solving the corresponding boundary-value problem, which fall above the diagram $1/z_*(\alpha)$ in Fig. 4, reflect stable stressed states of the material. Points which fall below this diagram correspond to unstable stressed states.

6. THE PLANE STRESSED STATE

In the general case of a plane stressed state ($\sigma_3 = 0$), by formula (3.4) we obtain

$$\frac{1}{z_*} = \text{sign}(\sigma_1)\frac{4 - 3m - 3m^2 + 4m^3}{4(1 - m + m^2)^{3/2}}, \quad m = \frac{\sigma_2}{\sigma_1} \tag{6.1}$$

The stability diagram, constructed for $\sigma_1 > 0$ using Eq. (6.1) as the continuous curve in Fig. 5, has a single zero at the point $m = -1$ which corresponds to simple shear, two asymptotes $1/z_* = \pm 1$ and, also, three extrema at points with coordinates: $m = (11 - \sqrt{105})/4 = 0.188$, $1/z_* = 25/6\sqrt{15}) = 1.08$; $m = 1$, $1/z_* = 0.5$; $m = (11 + \sqrt{105})/4 = 5.31$ and $1/z_* = 1.08$. The point with coordinates $m = 0$ and $1/z_* = 1$, corresponding to uniaxial extension and the point $(0; -1)$ reflecting uniaxial compression, are also characteristic. We also note the existence of the identity

$$1/z_*\,(1/m) = \text{sign}(m)\; 1/z_*(m)$$

In accordance with the physical meaning, the last equality means that the redesignation of $m = \sigma_2/\sigma_1$ by $1/m = \sigma_1/\sigma_2$ should have no effect on the quantity $1/z_*$ (when $m > 0$).

From the point of view of the stability loss, the maximum values $1/z_* = 1.08$ which are attained under conditions of biaxial extension with the relations between the stresses $\sigma_2 = 0.188\,\sigma_1$ and $\sigma_2 = 5.31\,\sigma_1$ are dangerous. In the domain of biaxial extension, the state of the material accompanying uniform biaxial

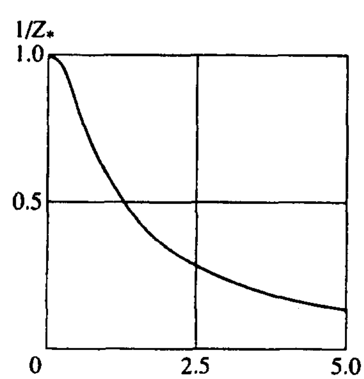

Fig. 4

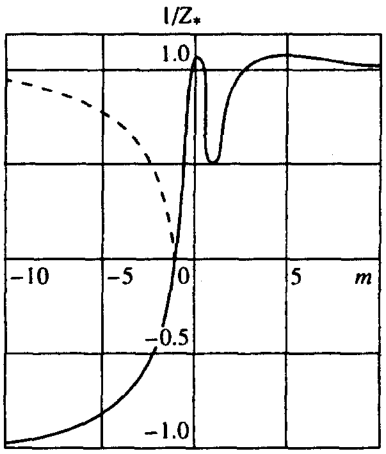

Fig. 5

extension ($m = 1$) turns out to be the most stable, and the value $1/z_* = 1$ is attained, in addition to the case of uniaxial extension, in the case of the two further stressed states $\sigma_2 = 0.368\sigma_1$ and $\sigma_2 = 2.72\sigma_1$.

When $\sigma_1 < 0$, the stability diagram of a stressed state is the mirror image, in the horizontal axis, of the diagram depicted by the continuous curve in Fig. 5. The part of the diagram when $\sigma_1 < 0$ in the domain $1/z_* > 0$ is shown by the dashed curve in Fig. 5.

Note that biaxial compressions, when $1/z > 0$, are found to be stable stressed states ($1/z_* < 0$).

The loading parameter $\alpha = \tau/\sigma$ was introduced in the case of the simplified plane stressed state (Fig. 3). It can be seen that this parameter is related to the parameter $m = \sigma_2/\sigma_1$ by the formulae

$$m = \frac{1 - \sqrt{1 + 4\alpha^2}}{1 + \sqrt{1 + 4\alpha^2}} \quad \text{and} \quad \alpha = \frac{1}{2}\sqrt{\left(\frac{1 - m}{1 + m}\right)^2 - 1}$$

In particular, when $\alpha \geqslant 0$ (Fig. 4), the interval $-1 \leqslant m \leqslant 0$ is the domain of variation of m.

7. THE PLANE DEFORMED STATE

In the case of plane deformation $\sigma_3 = (\sigma_1 + \sigma_2)/2$. Then, by formula (3.4), we have

$$\frac{1}{z_*} = \frac{\sqrt{3}}{2}\frac{\sigma_1 + \sigma_2}{|\sigma_1 - \sigma_2|} = \frac{\sqrt{3}\sigma_1}{2|\sigma_1|}\frac{1 + m}{|1 - m|} \tag{7.1}$$

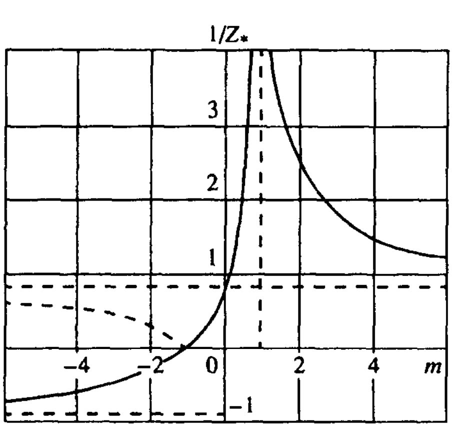

Fig. 6

The stability diagram when $\sigma_1 > 0$ has been constructed with the continuous curves in Fig. 6. The vertical asymptote corresponds to hydrostatic extension ($m = 1$) which is an unstable stressed state.

In the special case when $m = 0$, the stability parameter of the stressed state is equal to $\sqrt{3}/2 = 0.866$ and the stressed state is plane ($\sigma_2 = 0, \sigma_3 = \sigma_1/2$). The dashed line passing through the point with the coordinates (0, 0.866) is the horizontal asymptote of the stability diagram. The quantity $1/z_*$ at the above-mentioned point can also be determined using the diagram $1/z_*(m)$ for a plane stressed state (Fig. 5, $m = 0.5$).

The point with coordinates $m = -1$ and $1/z_* = 0$ reflects simple shear, and a second horizontal asymptote passes through the point (0, −0.866) which corresponds to biaxial compression ($\sigma_2 = 0$, $\sigma_3 = \sigma_1/2$, $\sigma_1 < 0$).

In the case when $\sigma_1 < 0$, the diagram for the change in the stability parameter is the mirror image of the diagram constructed with the continuous curves in Fig. 6. The part of the diagram when $\sigma_1 < 0$ in the domain of positive values of $1/z_*$ is shown by the dashed curve.

When $1/z > 0$, any triaxial compressions turn out to be stable stressed states of the material ($1/z_* < 0$), which does not contradict Bridgman's experimental data [6] for large plastic deformations of hollow thin-walled steel cylinders under an external pressure.

A plane state of strain can also be analysed using a second loading parameter, that is, $\beta = \tau_{max}/\sigma_0$, where τ_{max} is the maximum shear stress.

Substituting the equalities [8] $\sigma_1 = \sigma_0 + \tau_{max}$, $\sigma_2 = \sigma_0 - \tau_{max}$ into the right-hand side of Eq. (7.1), we obtain

$$\frac{1}{z_*} = \frac{\sqrt{3}}{2}\frac{\sigma_1 + \sigma_2}{|\sigma_1 - \sigma_2|} = \frac{\sqrt{3}}{2\beta}$$

The corresponding stability diagram, which consists of two branches of a hyperbola, is shown in Fig. 7. When $\beta \to \pm\infty$, the stressed state approximates to simple shear ($1/z_* \to 0$). The vertical asymptote in the domain $\beta > 0$ represents hydrostatic extension and, in the case when $\beta < 0$, hydrostatic compression, which is a stable stressed state.

Hence, the use of Drucker's stability postulate enables one to classify different stressed states using the stability parameter $1/z_*$.

8. THE RELATION BETWEEN THE STABILITY PARAMETER OF A STRESSED STATE AND THE OTHER PARAMETERS OF THE STRESSED STATE

We transform formula (3.4) to the form

$$\frac{1}{z_*} = \frac{\sqrt{3}}{4}\left\{3\frac{I_3(S_{ij})}{[I_2(S_{ij})]^{3/2}} + \frac{2}{3}\frac{I_1(\sigma_{ij})}{[I_2(S_{ij})]^{1/2}}\right\} \tag{8.1}$$

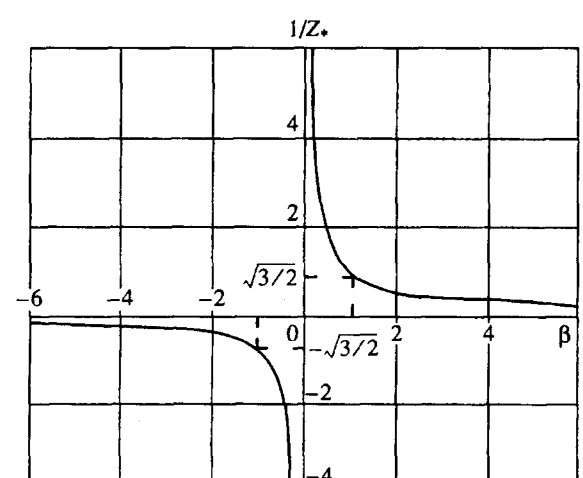

Fig. 7

In the theory of the pressure processing of metals, a number of different plasticity indicators, that is, parameters characterizing the experimental data on the plastic limit of metals, is introduced. In particular, the stiffness [9] $\Pi = 3\sigma_0/\sigma_e$ is one of the basic indicators.

Moreover, we have the relation [8]

$$27 I_3(S_{ij})/(2\sigma_e^3) = \cos\psi_\sigma$$

where ψ_σ is the angle of the form of the stressed state associated with the Nadai-Lode relation

$$\chi_\sigma = -\sqrt{3}\,\mathrm{ctg}(\psi_\sigma + \frac{4}{3}\pi)$$

Consequently, by formula (8.1),

$$1/z_* = (\Pi + \cos 3\psi_\sigma)/2$$

Note that, in the special case when $I_3(S_{ij}) = 0$, $\cos 3\psi_\sigma = 0$ holds, for example, in a plane deformed state and, therefore, $1/z_* = \Pi/2 = \sqrt{3}/(2\beta)$, where, as previously, $\beta = \tau_{max}/\sigma_0$.

Hence, the relation between the stability parameter of a stressed state $1/z_*$, which has a definite physical meaning, and the other fundamental parameters of a stressed state. II and χ_σ has been revealed.

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